Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Approximation Theory 135 (2005) 70–104
Journal of Approximation Theory
www.elsevier.com/locate/jat

Simple and double eigenvalues of the Hill operator with a two-term potential

Plamen Djakov[a,1], Boris Mityagin[b,*]

[a]Department of Mathematics, Sofia University, 1164 Sofia, Bulgaria
[b]Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210, USA

Received 15 June 2004; received in revised form 20 February 2005; accepted 15 March 2005

Communicated by Leonid Golinskii
Available online 3 May 2005

Abstract

We give a complete description of the structure of the spectra of Hill operator

$$Ly = -y'' + (a\cos 2x + b\cos 4x)y, \quad a, b \ \text{ real}, \quad x \in [0, \pi]$$

with periodic or antiperiodic boundary conditions. As in Ince [Proc. London Math. Soc. 23 (1923) 56] and Magnus–Winkler [Hill's Equation, Interscience Publishers, Wiley, 1969], properties and spectra of special tridiagonal matrices is a core of our analysis.

Keywords: Hill operator; Two-term potential; Coexistence problem

1. Introduction

The Schrödinger operator, considered on $\mathbb{R}$,

$$Ly = -y'' + v(x)y, \tag{1.1}$$

* Corresponding author.
E-mail addresses: djakov@fmi.uni-sofia.bg (P. Djakov), mityagin.1@osu.edu (B. Mityagin).
[1] Partially supported by Grant MM-1401/04 of the Bulgarian Ministry of Education and Science.

0021-9045/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2005.03.004

with a real-valued periodic potential $v(x) \in L^2([0,\pi])$, $v(x+\pi) = v(x)$, has spectral gaps, or instability zones $(\lambda_n^-, \lambda_n^+)$, $n \geqslant 1$, close to n^2 if n is large enough. The points λ_n^-, λ_n^+ could be determined as eigenvalues of the Hill operator

$$Ly \equiv -y'' + v(x)y, \tag{1.2}$$

considered on $[0,\pi]$ with boundary conditions

$$Per^+: \quad y(0) = y(\pi), \quad y'(0) = y'(\pi) \tag{1.3}$$

for even n, and

$$Per^-: \quad y(0) = -y(\pi), \quad y'(0) = -y'(\pi) \tag{1.4}$$

for odd n. See basics and details in [26,32,27,24,46].

The rate of decay of the sequence of spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ is closely related to the smoothness of the corresponding potential v. We will mention now only the Hochshtadt's result [18] that an $L^2([0,\pi])$-potential v is in C^∞ if and only if (γ_n) decays faster than any power of $(1/n)$. See the latest results and further references in [7,9].

In the case of specific potentials, like the Mathieu potential

$$v(x) = 2a\cos 2x, \tag{1.5}$$

or a more general two-term potential

$$v(x) = a\cos 2x + b\cos 4x, \tag{1.6}$$

general problems lead us to two classes of questions:

(i) Is the nth zone closed, i.e.,

$$\gamma_n = \lambda_n^+ - \lambda_n^- = 0, \tag{1.7}$$

or, equivalently, is the multiplicity of λ_n^+ equal to 2?

(ii) If $\gamma_n \neq 0$, could we tell more about the size of this gap, or, for large enough n, what is the asymptotic behavior of $\gamma_n = \gamma_n(v)$?

Question (i) for potential (1.5) was answered in a negative way by Ince [20]: the Mathieu–Hill operator has only *simple* eigenvalues both for Per^+ and Per^- boundary conditions, i.e., all zones of instability of the Mathieu–Schrödinger operator are open. His proof is presented in [13]. See other proofs of this fact in [17,28,29].

Question (ii) for the Mathieu potential was solved by Harrell [16] and Avron and Simon [2]. They showed for $v \in (1.5)$ that

$$\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8|a|^n}{[(n-1)!]^2}\left(1 + O(1/n^2)\right). \tag{1.8}$$

Earlier, Levi and Keller [25] gave asymptotics of the sequence $\gamma_n = \gamma_n(a)$ for $a \to 0$ when n is fixed. The question about the asymptotics of (γ_n) in the case of a two-term potential (1.6) was raised in [2], but remained unsolved. We found such asymptotics both for small a and b (when n is fixed), and for large n when a and b are fixed. First we have done it (see [10]) in the case when $8b = -a^2$. This led us to a proper understanding of the special parametrization of the coefficients a and b in (1.6) which comes from Whittaker's [47] and Ince's [21] analysis of this Hill operator. Further details could be found in Magnus–Winkler [48,26].

Put for real $a, b \neq 0$

$$a = -4\alpha t, \quad b = -2\alpha^2, \tag{1.9}$$

where either both α and t are real (if $b < 0$), or both are pure imaginary (if $b > 0$).

We show in [11,12] that the following asymptotic formulae hold for fixed α, t and $n \to \infty$: for even n

$$\gamma_n = \frac{8|\alpha|^n}{2^n[(n-2)!!]^2} \left|\cos\left(\frac{\pi}{2}t\right)\right| \left[1 + O((\log n)/n)\right] \tag{1.10}$$

and for odd n

$$\gamma_n = \frac{8|\alpha|^n}{2^n[(n-2)!!]^2} \frac{2}{\pi} \left|\sin\left(\frac{\pi}{2}t\right)\right| \left[1 + O((\log n)/n)\right], \tag{1.11}$$

where

$$(2m-1)!! = 1 \cdot 3 \cdots (2m-1), \qquad (2m)!! = 2 \cdot 4 \cdots (2m).$$

Proof, with all details, is given in [12]. It is based, on one hand, on our analytic methods developed in [7–9], and on the other hand, on the Ince's approach [20–22] approach [48,26] to coexistence problem (see (i) above) in the case of potential (1.6). More about Ince's gauge transform (2.7)—see Arscott [1], Urwin and Arscott [41], and Magnus and Winkler [26, Chapter 7].

We need to present (and this is done in this paper) their results in an appropriate form that serves our goal of finding asymptotics (1.10) and (1.11), or Theorems 1 and 3 in [11]. At the same time we sharpen their results about the multiplicities of the eigenvalues of the operator (1.2) + (1.6) in the case where t is an integer.

Finally, we give a complete description of the structure of the spectra of this operator, with full information about mutual positions of eigenvalues λ_n^-, λ_n^+ for Per^+ and Per^- boundary conditions in Theorem 11.

Volkmer [44] considered the general Ince equation

$$(1 + a\cos 2t)y'' + B(\sin 2t)y' + (c + d\cos 2t)y = 0,$$

where a, b, c, d are real, and $|a| < 1$. In the framework of Ince–Magnus–Winkler approach, he gave [44, Theorems 3 and 4] a solution of the coexistence problem, with detailed information on positions of eigenvalues corresponding to even and odd eigenfunctions.

Our Theorem 11 could be derived from Theorem 3 or Eqs. (27), (28) in [44].

2. Preliminaries on Ince method and the Hill operator (1.6)

In this section, we present in a convenient form for our further analysis the results of Ince [20–22] and Magnus and Winkler [48,26]. Then we go further into a careful and detailed analysis of first open gaps when the series of even (or odd) gaps has only finitely many open ones.

1. A potential, or a family of two-term potentials

$$v(x) = a \sin 2x + b \cos 4x, \qquad a, b \text{ real} \tag{2.1}$$

and the question about asymptotics of spectral gaps, or zones of instability, of corresponding Schrödinger operator

$$Ly = -y'' + v(x)y, \quad -\infty < x < +\infty, \tag{2.2}$$

has been discussed in [2,15,10] but until recently the sharp asymptotics of spectral gaps has not been known. We found such an asymptotics; see Theorems 1 and 3 in [11], and details in [12].

Notice that we change the potential, or the entire operator L, by using elementary transformations in such a way that the spectrum is preserved both for the Schrödinger operator, and for the Hill operator, considered with Per^+ or Per^- boundary conditions.

(a) A shift of x to $x + \pi/2$ changes $v \in$ (2.1) to

$$v_1(x) = -a \sin 2x + b \cos 4x. \tag{2.3}$$

It implies that without loss of generality in our analysis of spectra of $L_v = L \in$ (2.2) we can assume that $a > 0$ (or, $a < 0$ if we would prefer).

(b) A shift of x to $x + \pi/4$ changes $v_1 \in$ (2.3) to

$$v_2(x) = -a \cos 2x - b \cos 4x. \tag{2.4}$$

Let us use this form (2.4) to make the most important transformation which annihilates the term with higher frequency. (See further comments in Section 5.1).

(c) Put

$$K = E^{-1} L E, \tag{2.5}$$

where

$$Ly = -y'' + v_2(x)y, \tag{2.6}$$

$$Eu = u \exp(\alpha \cos 2x), \tag{2.7}$$

$$y = u \exp(\alpha \cos 2x). \tag{2.8}$$

Then

$$-E^{-1} L E u = u'' - 4\alpha(\sin 2x)u' + (2\alpha^2 + (a - 4\alpha)\cos 2x + (b - 2\alpha^2)\cos 4x)u \tag{2.9}$$

and if we choose α so that

$$2\alpha^2 = b \tag{2.10}$$

then

$$\begin{aligned}(K - \lambda)u &= E^{-1}(L - \lambda)Eu \\ &= -u'' + 4\alpha(\sin 2x)u' - (\lambda + 2\alpha^2 + (a - 4\alpha)\cos 2x)u.\end{aligned} \tag{2.11}$$

The operator K, with any choice of a complex number α, is similar to L, so

$$\sigma(K) = \sigma(L), \tag{2.12}$$

although K is not necessarily self-adjoint as L was. K is self-adjoint if

$$\alpha = i\tau, \qquad \tau \text{ real.} \tag{2.13}$$

But K has at least two nice features.

(i) Its potential does not have terms of high-frequency $\cos 4x$ and $\sin 4x$.
(ii) With an even coefficient for u and an odd coefficient for u', the subspaces of even functions and odd functions are invariant for K. Therefore, K can be considered as a direct sum of two simpler operators K^{odd} and K^{even}, with $\sigma(K)$ being a union of the spectra of these operators.

We make this vague remark (ii) more precise in analysis of the Hill operator K with $Per^{\pm}$ boundary conditions.
2. Now we consider K on $[0, \pi]$ with boundary conditions

$$Per^{+}: \quad u(0) = u(\pi), \quad u'(0) = u'(\pi), \tag{2.14}$$

or

$$Per^{-}: \quad u(0) = -u(\pi), \quad u'(0) = -u'(\pi). \tag{2.15}$$

Two linearly independent eigenfunctions cannot be even (or odd) simultaneously; therefore, if w is an eigenfunction of K (in either case $Per^{\pm}$) then its even and odd parts are eigenfunctions as well

$$w^{\pm}(x) = \frac{1}{2}(w(x) \pm w(-x)). \tag{2.16}$$

Therefore, if K has *two* $Per^{\pm}$ linearly independent *λ-eigenfunctions*, i.e.,

$$Kw = \lambda w, \quad w \in L^2 \quad \text{for} \quad Per^{\pm}, \tag{2.17}$$

then we have *one* even nonzero solution $w_0 = w^{+}$, and *one* odd nonzero solution $w_1 = w^{-}$. Then

$$w_0(x) = \sum_{n \in \Gamma} A_n \cos nx, \tag{2.18}$$

$$w_1(x) = \sum_{n \in \Gamma} B_n \sin nx, \tag{2.19}$$

with

$$\Gamma = 2\mathbb{Z}_+ = \{0\} \cup 2\mathbb{N} \quad \text{for } Per^{+}. \tag{2.20}$$

$$\Gamma = 2\mathbb{Z}_+ + 1 = 2\mathbb{N} - 1 \quad \text{for } Per^{-}. \tag{2.21}$$

Put

$$\lambda + 2\alpha^2 = \lambda + b = \mu \tag{2.22}$$

and

$$a = 4\alpha t, \quad \text{so} \quad a - 4\alpha = 4\alpha(t-1). \tag{2.23}$$

Now a direct substitution shows that

$$(K - \lambda)w = 0 \tag{2.24}$$

can be rewritten in the following way:

Case Per^+: Then by (2.18)

$$w_0(x) = A_0 + \sum_{k \in 2\mathbb{N}} A_k \cos kx, \tag{2.25}$$

$$w_1(x) = \sum_{k \in 2\mathbb{N}} B_k \sin kx \tag{2.26}$$

and Eq. (2.24) for (2.25) is equivalent to the system (k even)

$$-\mu A_0 + 2\alpha(t-1)A_2 = 0, \tag{2.27}$$

$$4\alpha(t+1)A_0 + (2^2 - \mu)A_2 + 2\alpha(t-3)A_4 = 0, \tag{2.28}$$

$$2\alpha(t-1+k)A_{k-2} + (k^2 - \mu)A_k + 2\alpha(t-1-k)A_{k+2} = 0, \quad k \geqslant 4. \tag{2.29}$$

[In [26] in line (7.17), $n = 1$, p. 95, corresponding to (2.28), the coefficient 2 is written although 4 is correct.]

Respectively, for (2.26) Eq. (2.24) is equivalent to the system

$$(2^2 - \mu)B_2 + 2\alpha(t-3)B_4 = 0, \tag{2.30}$$

$$2\alpha(t-1+k)B_{k-2} + (k^2 - \mu)B_k + 2\alpha(t-1-k)B_{k+2} = 0, \quad k \geqslant 4. \tag{2.31}$$

Case Per^-: Then we have

$$w_0(x) = \sum_{k \in 2\mathbb{N}-1} A_k \cos kx, \tag{2.32}$$

$$w_1(x) = \sum_{k \in 2\mathbb{N}-1} B_k \sin kx. \tag{2.33}$$

For (2.32) Eq. (2.24) is equivalent to the system (k odd)

$$(1 - \mu + 2\alpha t)A_1 + 2\alpha(t-2)A_3 = 0, \tag{2.34}$$

$$2\alpha(t-1+k)A_{k-2} + (k^2 - \mu)A_k + 2\alpha(t-1-k)A_{k+2} = 0, \quad k \geqslant 3. \tag{2.35}$$

Respectively, (2.24) for (2.33) leads to the system (k odd)

$$(1-\mu-2\alpha t)B_1 + 2\alpha(t-2)B_3 = 0, \tag{2.36}$$

$$2\alpha(t-1+k)B_{k-2} + (k^2-\mu)B_k + 2\alpha(t-1-k)B_{k+2} = 0, \quad k \geqslant 3. \tag{2.37}$$

3. In the case of Mathieu operator (the recurrence system is simpler there) Ince [20] explained that all gaps are open, i.e., all eigenvalues are simple, by considering a discrete Wronskian. In the case of the operator K its analog would be the sequence

$$\Delta_k = \begin{vmatrix} A_k & A_{k+2} \\ B_k & B_{k+2} \end{vmatrix}, \qquad k \in \Gamma, \tag{2.38}$$

where Γ means evens for Per^+ and odds for Per^-. For Per^+ we have, if t is not odd, that

$$\begin{aligned} &A_0 = 1, \quad A_2 = \tfrac{\mu}{2\alpha(t-1)}, \quad A_4 = \tfrac{\mu(\mu-4)}{4\alpha^2(t-1)(t-3)} - 2\tfrac{t+1}{t-3}, \\ &B_0 = 0, \quad B_2 = 1, \qquad\quad B_4 = \tfrac{\mu-4}{2\alpha(t-3)} \end{aligned} \tag{2.39}$$

and therefore,

$$\Delta_0 = 1, \quad \Delta_2 = 2\frac{t+1}{t-3}. \tag{2.40}$$

For Per^-, if t is not even, then

$$\begin{aligned} &A_1 = 1, \ A_3 = (\mu-1-2\alpha t)/2\alpha(t-2), \\ &B_1 = 1, \ B_3 = (\mu-1+2\alpha t)/2\alpha(t-2) \end{aligned} \tag{2.41}$$

and

$$\Delta_1 = \frac{2t}{t-2}. \tag{2.42}$$

Notice, that Eqs. (2.29) and (2.31), or (2.35) and (2.37) are identical (but k is odd or even). Let us compare A- and B-solutions in Per^+-case, i.e., when (2.29) and (2.31) hold. Multiply (2.29) by B_k and (2.31) by A_k and subtract these identities; we get

$$2\alpha(t-1+k)\Delta_{k-2} - 2\alpha(t-1-k)\Delta_k = 0, \quad k \geqslant 4, \tag{2.43}$$

or

$$\Delta_k = \frac{t-1+k}{t-1-k}\Delta_{k-2} \quad k \ \text{ even}, \ k \geqslant 4. \tag{2.44}$$

In Per^- case, by manipulating (2.35) and (2.37), one comes to the recurrence

$$\Delta_k = -\frac{k+(t-1)}{k-(t-1)}\Delta_{k-2}, \qquad k \ \text{ odd}, \ k \geqslant 3. \tag{2.45}$$

If $A = (A_k)_{k \in \Gamma}$ and $B = (B_k)_{k \in \Gamma}$ are ℓ^2-solutions of (2.29) and (2.31) correspondingly [or, of (2.35) and (2.37)], then by dividing (2.29) and (2.31), and (2.35) and (2.37) by $k^2 - \mu$ we get

$$|A_k| + |B_k| = \frac{1}{k}\xi_k, \quad (\xi_k) \in \ell^2$$

and by (2.38)

$$\lim_{k\to\infty} k^2 |\Delta_k| = 0. \tag{2.46}$$

But for any $m \in \Gamma$, by (2.44) or (2.45),

$$\Delta_{m+2p} = (-1)^p \left(\prod_{j=1}^{p} \frac{m + 2j + (t-1)}{m + 2j - (t-1)} \right) \cdot \Delta_m. \tag{2.47}$$

If $t \geqslant 0$, and $m, j > 0$

$$\frac{m + 2j + t - 1}{m + 2j - (t-1)} \geqslant \frac{m + 2j - 1}{m + (2j-1) + 2} \tag{2.48}$$

so

$$\prod_{j=1}^{p} \frac{m + 2j + (t-1)}{m + 2j - (t-1)} \geqslant \frac{m+1}{m+2p+1} \tag{2.49}$$

and for any p

$$|\Delta_{m+2p}|(m + 2p + 1) \geqslant |\Delta_m|(m+1). \tag{2.50}$$

Now (2.46) implies that

$$\Delta_m \equiv 0. \tag{2.51}$$

However, this fact and our evaluation in (2.40) [and (2.42)] show the following:

(a) If t is not an odd positive integer and solution (2.39) of (2.27)–(2.29) and (2.30)–(2.31) lies in ℓ^2 then

$$\Delta_2 = 2\frac{t+1}{t-3} \neq 0 \quad \text{and} \quad \Delta_2 = 0. \tag{2.52}$$

(b) If t is not an even positive integer and solutions (2.41) of (2.34)–(2.35) and (2.36)–(2.37) happen to be in ℓ^2 then

$$\Delta_1 = \frac{2t}{t-2} \neq 0 \quad \text{and} \quad \Delta_1 = 0. \tag{2.53}$$

These contradictions prove the following (See [26, Theorem 7.9]).

Proposition 1. *Consider the operator*

$$Ly = -y'' + (a\cos 2x - b\cos 4x), \tag{2.54}$$

where

$$a^2 = 8bt^2, \quad t > 0. \tag{2.55}$$

(i) *If t is not odd, then all eigenvalues of L with $bc = Per^+$ are simple, so all even zones of instability are open.*
(ii) *If t is not even, then all eigenvalues of L with $bc = Per^-$ are simple, so all odd zones of instability are open.*

In conclusion of this section, let us notice that the assumption $b > 0$ [see (2.10) or (2.55)] in Proposition 1 can be omitted. If $b < 0$ then (2.10) leads to a pure imaginary α, and (2.55) gives a pure imaginary $t \neq 0$. All constructions and arguments remain valid; even the operator

$$K(\alpha) = \exp(-\alpha\cos 2x)L(a,b)\exp(\alpha\cos 2x), \tag{2.56}$$

where $L(a,b) \in (2.2) + (2.1)$ is self-adjoint in this case.

If t is pure imaginary, say, $t = is$, then

$$\left|\frac{m+2j-1+is}{m+2j+1-is}\right| = \left(\frac{(m+2j-1)^2+s^2}{(m+2j+1)^2+s^2}\right)^{1/2} \geqslant \frac{m+2j-1}{m+2j+1}$$

and as in (2.48), (2.49) we come to inequality (2.50) in the case where $t = is$, s real.

Therefore, we have

Proposition 2. *If $b < 0$ and t is pure imaginary in* (2.55), *then all eigenvalues of $L \in$* (2.54) *with $bc = Per^+$ or Per^- are simple, so all zones of instability are open.*

We analyze the spectra $\sigma(L_{Per^\pm})$ in the case where t is a positive integer in the next section. However, let us notice that the assumption $t > 0$ is not a restriction because t and $-t$ give a rise to isospectral operators.

3. Case $a = -4\alpha t$, $b = -2\alpha^2$

0. In this section we consider potentials (1.6), i.e., $v(x) = a\cos 2x + b\cos 4x$. Therefore, we put $b = -2\alpha^2$ to fit to the previous section, where we consider potentials in form (2.4) or (2.54), with $-b$ in front of $\cos 4x$. There we considered an operator $K \in (2.5)$ [or (2.56)] similar to L if

$$2\alpha^2 = b \tag{3.1}$$

and analyzed its spectrum by using its decomposition into even and odd parts K^{even}, K^{odd} and then dealing with matrix representations of these components. These matrices, or recurrences (2.27)–(2.37) will be used in this section as well to get more information in the case where

$$a^2 = 8bt^2, \quad t \in \mathbb{N}. \tag{3.2}$$

Of course, in view of Proposition 1, if t is even, respectively odd, we need to analyze $\sigma(L_{Per^-})$, respectively $\sigma(L_{Per^+})$.

1. In either case the following elementary lemma will be useful.

Lemma 3. *Suppose $D = (D_{ij})_0^n$ is a three-diagonal matrix of the form*

$$D = \begin{bmatrix} d_0 & p_0 & 0 & 0 & & & & & & \\ q_1 & d_1 & p_1 & 0 & 0 & & & & & \\ 0 & q_2 & d_2 & p_2 & 0 & 0 & & & & \\ 0 & 0 & q_3 & d_3 & p_3 & 0 & 0 & & & \\ & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & 0 & q_{n-2} & d_{n-2} & p_{n-2} & 0 \\ & & & & & & 0 & q_{n-1} & d_{n-1} & p_{n-1} \\ & & & & & & 0 & 0 & q_n & d_n \end{bmatrix} \tag{3.3}$$

with

$$p_0, \dots, p_{n-1} \neq 0, \qquad q_1, \dots, q_n \neq 0. \tag{3.4}$$

With fixed n, denote

$$D^k = \left(D_{ij}\right)_{i,j=k}^n \tag{3.5}$$

and

$$\delta^k = \det\, D^k, \qquad k = 0, 1, \dots, n. \tag{3.6}$$

Then

$$|\delta^0| + |\delta^1| > 0, \tag{3.7}$$

i.e., the determinants δ^0 and δ^1 could not be zeroes simultaneously.

Proof. If $n = 1$ then

$$\delta^1 = d_1, \qquad \delta^0 = d_0 d_1 - p_0 q_1. \tag{3.8}$$

If $d_1 \neq 0$ then (3.7) holds. But if $d_1 = 0$ then $\delta^0 = -p_0 q_1 \neq 0$ by (3.4), and (3.7) holds as well.

Now we proceed by induction by n (recall that D is $(n+1) \times (n+1)$-matrix). By (3.3)

$$\delta^0 = d_0 \delta^1 - p_0 q_1 \delta^2. \tag{3.9}$$

If (3.7) does not hold, i.e., $\delta^0 = \delta^1 = 0$, then with $p_0 q_1 \neq 0$ (3.9) implies $\delta^2 = 0$. Then $\delta^1 = \delta^2 = 0$, and D^1 is $n \times n$ matrix, which leads us to a contradiction. $\square$

For Per^- case we need an analogue of Lemma 3.

Lemma 4. *Consider two* 3-*diagonal* $n \times n$ *matrices*

$$D^{\pm} = \begin{bmatrix} d_1 \pm d & p_1 & 0 & 0 & & & & \\ q_2 & d_2 & p_2 & 0 & 0 & & & \\ 0 & q_3 & d_3 & p_3 & 0 & 0 & & \\ & & & & & & & \\ . & . & . & . & . & . & . & . \\ & & & 0 & q_{n-2} & d_{n-2} & p_{n-2} & 0 \\ & & & & 0 & q_{n-1} & d_{n-1} & p_{n-1} \\ & & & & 0 & 0 & q_n & d_n \end{bmatrix}, \tag{3.10}$$

where

$$p_1, \ldots, p_{n-1} \neq 0, \quad q_2, \ldots, q_n \neq 0 \ \text{ and } d \neq 0. \tag{3.11}$$

Put

$$\delta^{\pm} = \det\ D^{\pm}. \tag{3.12}$$

Then

$$|\delta^+| + |\delta^-| > 0, \tag{3.13}$$

i.e., the determinants δ^+ *and* δ^+ *could not be zeroes simultaneously*.

Proof. Decomposing along the first row, we obtain

$$\delta^{\pm} = (d_1 \pm d)\delta^2 - p_1 q_2 \delta^3 \tag{3.14}$$

so

$$2d\delta^2 = \delta^+ - \delta^-. \tag{3.15}$$

If $\delta^+ = \delta^- = 0$ then $\delta^2 = 0$ (because $d \neq 0$), and by (3.14) $\delta^3 = 0$ (because $p_1 q_2 \neq 0$). But this contradicts Lemma 1 if we apply it to the matrix D^2. $\square$

2. Let $t = 2p - 1,\ p \geqslant 1$. By Proposition 1 (ii), all eigenvalues of L_{Per^-} (and of K) are simple. Now we consider the case Per^+. The spectral equation (2.17) can be split into even and odd components; if $w = (A, B)$ then (2.17) becomes

$$(K^{\text{even}} - \lambda)A = 0, \quad (K^{\text{odd}} - \lambda)B = 0,$$

or in matrix form

$$(H^0 - \mu)A = 0, \qquad (H^2 - \mu)B = 0, \tag{3.16}$$

where (with k even) H^0 is

$$\begin{bmatrix} 0 & 2\alpha(t-1) & 0 & & & & \cdot \\ 4\alpha(t-1) & 2^2 & 2\alpha(t-3) & & & & \cdot \\ 0 & 2\alpha(t-3) & 4^2 & 2\alpha(t-5) & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 2\alpha(t-1+k) & k^2 & 2\alpha(t-1-k) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{3.17}$$

as it follows from (2.25)–(2.31).

All terms on the off-diagonals are nonzero but *one* on the pth line [see (2.29) or (2.31)]

$$t-1-k=0 \quad \text{if} \quad k=2p-2. \tag{3.18}$$

This partially decouples systems (2.27)–(2.29) and (2.30)–(2.31). If

$$A=(a,a'), \quad a=(A_k)_0^{2p-2}, \;\; a'=(A_k)_{2p}^{\infty}, \quad k \;\text{ even} \tag{3.19}$$

and the same for B, i.e.,

$$B=(b,b'), \quad b=(B_k)_2^{2p-2}, \;\; b'=(B_k)_{2p}^{\infty}, \quad k \;\text{ even}, \tag{3.20}$$

then

$$\begin{aligned} &(1) \qquad (H^0_{2p-2}-\mu)a=0, \\ &(2) \;\; a_{2p-2}\cdot 2\alpha\cdot 4(p-1)e_{2p}+(H^{2p}-\mu)a'=0, \end{aligned} \tag{3.21}$$

where $e_p=(\delta_{ip})_{i\in\Gamma}$ is a coordinate unit vector in ℓ^2, and

$$\begin{aligned} &(1) \qquad (H^2_{2p-2}-\mu)b=0, \\ &(2) \;\; b_{2p-2}\cdot 2\alpha\cdot 4(p-1)e_{2p}+(H^{2p}-\mu)b'=0. \end{aligned} \tag{3.22}$$

Lemma 5. *If μ is a Per^+ eigenvalue for K of multiplicity* 1, *then*

$$\delta^0(\mu;\alpha)=0 \quad \textit{or} \quad \delta^1(\mu;\alpha)=0. \tag{3.23}$$

Remark. With p fixed, we omit it in the notations of δ^0 and δ^1

$$\delta^0(\mu;\alpha)=\det\left(H^0_{2p-2}(\alpha)-\mu\right), \tag{3.24}$$

$$\delta^1(\mu;\alpha)=\det\left(H^2_{2p-2}(\alpha)-\mu\right). \tag{3.25}$$

Notice that

$$\deg\,\delta^0=p, \quad \deg\,\delta^1=p-1. \tag{3.26}$$

If $\alpha=0$ then

$$\delta^0(\mu;0)=-\mu\delta^1(\mu;0)=-\mu\prod_1^{p-1}[(2j)^2-\mu]. \tag{3.27}$$

Proof of Lemma 5. First, we assume $p \geqslant 2$. By (ii) in Section 2.1, if

$$Ku = \mu u, \quad u \neq 0 \tag{3.28}$$

and

$$\dim\, E(\mu) = 1, \tag{3.29}$$

then

(i) u is even but no odd nonzero function satisfies (3.28), or
(ii) u is odd but no even nonzero function satisfies (3.28).

In case (i)

$$u = A_0 + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} A_k \cos kx \tag{3.30}$$

and, with notations (3.19), Eq. (3.21) holds. We claim that

$$\delta^0(\mu, \alpha) = 0. \tag{3.31}$$

Otherwise, by (1) in (3.21), $a = 0$, its component $A_{2p-2} = 0$ as well, the second equation in (3.21) becomes just

$$(H^{2p} - \mu)a' = 0. \tag{3.32}$$

With $u \neq 0$ we should have $a' \neq 0$ as well. But Eqs. (3.21.2) and (3.22.2) are essentially the same, so if we define

$$B = (0, b'), \quad b' = a', \tag{3.33}$$

(see notations (3.20)) we get a sequence B such that (3.22) holds. It gives us a nonzero odd function

$$v(x) = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} A_k \sin kx \tag{3.34}$$

which satisfies (3.28), and therefore, the multiplicity of μ is $\geqslant 2$. This contradiction proves (3.31).

In case (ii)

$$u = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} B_k \sin kx, \quad v \neq 0 \tag{3.35}$$

and

$$Kv = \mu v. \tag{3.36}$$

We claim that

$$\delta^1(\mu; \alpha) = 0. \tag{3.37}$$

Otherwise, by (3.22.1) $b = 0$, and by (3.22.2)

$$(H^{2p} - \mu)b' = 0, \quad b' \neq 0.$$

Then

$$u = \sum_{\substack{k=2p \\ k \text{ even}}}^{\infty} B_k \cos kx, \quad v \neq 0 \tag{3.38}$$

is a nonzero even solution of (3.28). This contradiction proves (3.37). Lemma 5 is proven for $p \geqslant 2$.

If $p = 1$ then the matrix $H^0 \in (3.17)$ has the form

$$H = \begin{bmatrix} 0 & 0 & 0 & & \\ 8\alpha & 4 & -4\alpha & 0 & \\ 0 & 8\alpha & 16 & -8\alpha & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3.39}$$

and

$$\delta^0(\mu; \alpha) = -\mu \qquad \forall \alpha, \tag{3.40}$$

but an analogue of $\delta^1 \in (3.25)$ is not defined. We claim: If $\mu \neq 0$ is an eigenvalue of K_{Per^+} then its multiplicity is 2. Indeed, if $u \in (3.30) + (3.28)$ then (3.21.1) tells us that

$$-\mu A_0 = 0, \tag{3.41}$$

so $A_0 = 0$, and by (3.21.2)

$$(H^2 - \mu)a' = 0, \qquad a' \neq 0. \tag{3.42}$$

As in (3.32)–(3.33) it gives a second nonzero solution $v \in (3.34)$ of (3.28), so the multiplicity of μ is 2.

Vice versa, if $v \in (3.35)$ is a solution of (3.28) then

$$u = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} B_k \cos kx$$

is a nonzero solution of (3.28), and again the multiplicity is 2.

Therefore, if $p = 1$ and μ is of multiplicity 1, then $\mu = 0$, i.e., it is a root of the polynomial (3.40). Lemma 5 is proven. □

3. By Proposition 1(i), for an even $t = 2m,\ m \geqslant 1$, all eigenvalues of L_{Per^+} (and of the corresponding operator K) are simple. [See the comment related to complex α in Section 5.5.] So, we need to analyze only the case Per^-. Again, we decompose functions and K into even and odd components; if by (2.32)–(2.33) $w = (A; B)$ then (2.17) becomes $(K^{\text{even}} - \mu)A = 0$, $(K^{\text{odd}} - \mu)B = 0$, or in matrix form

$$(H^+ - \mu)A = 0, \qquad (H^- - \mu)B = 0, \tag{3.43}$$

where $A \in (2.32),\ B \in (2.33),\ \Gamma = 2\mathbb{N}$, and by (2.34)–(2.37), $k \in \Gamma$,

$$H^\pm = \begin{bmatrix} 1 \pm 2\alpha t & 2\alpha(t-2) & 0 & & \\ 2\alpha(t+2) & 3^2 & 2\alpha(t-4) & 0 & \\ & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots\ \cdots \\ \cdots & 2\alpha(t-1+k) & k^2 & 2\alpha(t-1-k) & 0\ \ \cdots \\ & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots\ \cdots \end{bmatrix}. \tag{3.44}$$

We do not repeat all the details which are essentially the same as in the previous subsection.

All terms on the off-diagonals are nonzero but one in the j_*th line, when $j_* = m$ as

$$t - 1 - (2j-1) = 0 \quad \text{if} \quad t = 2m,\ j = m. \tag{3.45}$$

Let $H_m^\pm$ be the left-upper $m \times m$ submatrix of $H^\pm$, and

$$\delta^\pm(\mu; \alpha) = \det\left(H_m^\pm - \mu\right). \tag{3.46}$$

Notice that [compare (3.26)] now

$$\deg\ \delta^+ = \deg\ \delta^- = m \geqslant 1 \tag{3.47}$$

in both cases, and if $\alpha = 0$

$$\delta^+(\mu; 0) = \delta^-(\mu; 0) = \prod_{j=1}^{m} \left[(2j-1)^2 - \mu\right]. \tag{3.48}$$

Now “heads” of A and B [compare (3.19), (3.20)]

$$a = (A_k)_1^{2m-1}, \quad b = (B_k)_1^{2m-1}, \quad k \text{ odd}, \tag{3.49}$$

have the same size (m-vectors), and “tails”

$$a' = (A_k)_{2m+1}^\infty, \quad b' = (B_k)_{2m+1}^\infty, \quad k \text{ odd}, \tag{3.50}$$

satisfy

$$X_{2m-1} \cdot 4\alpha(2m-1)e_{2m+1} + \left(H^{2m+1} - \mu\right)x' = 0, \tag{3.51}$$

where $x' = (X_k)_{2m+1}^{\infty}$, k is odd, and H^{2m+1} is a lower right infinite block of the matrix $H^{\pm}$ without m upper rows and m left columns.

Lemma 6. *If μ is a Per^- eigenvalue for K of multiplicity* 1 *then*

$$\delta^+(\mu, \alpha) = 0 \quad or \quad \delta^-(\mu, \alpha) = 0. \tag{3.52}$$

Proof. Would be a copy of the Lemma 5's proof and we omit it. Of course, Lemma 4 is used instead of Lemma 3.

4. Lemmas 5 and 6 already lead to conclusion that if t is an integer then all but maybe $[t/2]$ gaps are closed.

Proposition 7. (a) *If $t = 2p - 1$, $p \geqslant 1$, then the number of open even gaps does not exceed $p - 1$.*

(b) *If $t = 2m$, $m \geqslant 1$, then the number of open odd gaps does not exceed m.*

Proof. Each open gap $\{\lambda^-, \lambda^+\}$, or $\{\mu^-, \mu^+\}$, gives two simple eigenvalues of K_{Per^+} or K_{Per^-}. Such eigenvalues, by Lemmas 5 and 6, are among the roots

$$R^* = R^0 \cup R^1, \quad R^0 := \{\mu : \ \delta^0(\mu; \alpha) = 0\}, \quad R^1 := \{\mu : \ \delta^1(\mu; \alpha) = 0\} \tag{3.53}$$

for $\mu \in \sigma(Per^+)$, and

$$R_* = R^+ \cup R^-, \quad R^+ := \{\mu : \ \delta^+(\mu; \alpha) = 0\}, \quad R^- := \{\mu : \ \delta^-(\mu; \alpha) = 0\} \tag{3.54}$$

for $\mu \in \sigma(Per^-)$.

With $t = 2p - 1 \geqslant 1$, by (3.26),

$$\# R^* \leqslant p + (p - 1) = 2(p - 1) + 1 \tag{3.55}$$

and the number of pairs of simple eigenvalues does not exceed $p - 1$.

If $t = 2m$, $m \geqslant 1$, by (3.47),

$$\# R_* \leqslant m + m = 2m \tag{3.56}$$

and the number of pairs of simple eigenvalues does not exceed m. In both cases, this number is $\leqslant [t/2]$. □

4. Finitely many open even (odd) gaps

Proposition 7 gives some improvement of Theorem 7.9 in [26, p. 107], which claims the inequality $\leqslant [t/2] + 1$. But we want to get more information about the structure of these open gaps. In particular, we will explain that the number of those gaps *is equal* to $[t/2]$.

1. We need a few technical remarks on matrices H^{2p} (of (3.21)–(3.22)) and $H^{2m+1} \in$ (3.51). Lemmas 3 and 4 told something about *finite* tridiagonal matrices. Now consider

an infinite tridiagonal matrix h, $h = D + P + Q$, with D being diagonal and P, Q off-diagonals,

$$h = \begin{bmatrix} d_0 & p_0 & & & \\ q_1 & d_1 & p_1 & & \\ & q_2 & d_2 & p_2 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{4.1}$$

We assume that the following conditions hold:

$$d_k \in \mathbb{R}, \quad |d_k| \to \infty \quad (k \to \infty), \tag{4.2}$$

$$(|p_k| + |q_k|)/d_k \to 0, \tag{4.3}$$

$$p_k \neq 0,\ k = 0, 1, \ldots; \quad q_k \neq 0,\ k = 1, 2, \ldots . \tag{4.4}$$

Lemma 8. *The matrix h defines an operator in ℓ^2 which spectrum $\sigma(h)$ is discrete, and*

$$\sigma(h) = \{\mu_j\}_0^\infty, \quad \mu_j \to \infty \tag{4.5}$$

and each $\mu = \mu_j \in \sigma(h)$ is an eigenvalue of geometric multiplicity 1.

Proof. Condition (4.3) guarantees that for large enough $r > 0$

$$\sup_{0 \leqslant k < \infty} 2 \cdot \frac{|p_k| + |q_k|}{r + |d_k|} \leqslant \frac{1}{2}. \tag{4.6}$$

Indeed, there exists $k_* < \infty$ such that

$$\frac{|p_k| + |q_k|}{|d_k|} \leqslant \frac{1}{4} \quad \text{for} \ \ k \geqslant k_*. \tag{4.7}$$

Define

$$r_* = 1 + 4 \sup\{|p_k| + |q_k| \ : \ 0 \leqslant k \leqslant k_*\}; \tag{4.8}$$

then (4.6) holds for $r \geqslant r_*$. Put

$$z = ir, \qquad r \geqslant r_*. \tag{4.9}$$

Then $f = (z - h)^{-1}$ is well defined. Indeed, see (4.1),

$$\begin{aligned} &z - h = (z - D) - (P + Q) = (z - D)(1 - T), \\ &T := (z - D)^{-1}(P + Q), \end{aligned} \tag{4.10}$$

where $z - D$ is a diagonal operator with diagonal terms

$$z - d_k, \quad |z - d_k| = (r^2 + |d_k|^2)^{1/2} \geqslant (r + |d_k|)/2. \tag{4.11}$$

Now (4.6) implies that

$$\|T\| = \|(z - D)^{-1}(P + Q)\| \leqslant 1/2 \tag{4.12}$$

and therefore,

$$(z-h)^{-1} = (1-T)^{-1}(z-D)^{-1} \tag{4.13}$$

is well defined, and $\|(1-T)^{-1}\| \leqslant 2$. In view of (4.11) and (4.2), the operator $(z-D)^{-1}$ is compact, thus $(z-h)^{-1}$ is compact also. By the Riesz Theorem its spectrum is a sequence $\{\alpha_j\}$ such that $\alpha_j \to 0$, and therefore,

$$\sigma(h) = \{\mu_j\}, \quad \mu_j = z - 1/\alpha_j \to \infty. \tag{4.14}$$

Moreover, the projectors

$$P_j = \frac{1}{2\pi i}\int_{C_j} (\zeta - h)^{-1} d\zeta, \tag{4.15}$$

where

$$C_j = \{\zeta \in \mathbb{C} : \ |\zeta - \mu_j| = \delta_j, \quad \delta_j = \frac{1}{2}\min_{\tilde{j}\neq j} |\mu_j - \mu_{\tilde{j}}|,$$

are finite-dimensional.

There is only *one* eigenvector $g = g_j$ with an eigenvalue $\mu = \mu_j$ as it follows from (4.1) and (4.4). Indeed, there is only one sequence $x = (x_k)_0^\infty$, even without the restriction to be in ℓ^2, which satisfies $(h-\mu)x = 0$, or recurrences

$$d_0 x_0 + p_0 x_1 = 0,$$

$$q_1 x_0 + d_1 x_1 + p_1 x_2 = 0$$

and so on. If $x_0 = \tau$, then (with $p_k \neq 0$),

$$x_1 = \frac{d_0}{p_0}\tau, \quad x_{k+1} = -\frac{1}{p_k}(q_k x_{k-1} + d_k x_k). \tag{4.16}$$

It means that [geometric] multiplicity of μ is 1. Lemma 8 is proven. □

2. Now we are ready to prove the following.

Lemma 9. *For each real* $\alpha \neq 0$;

(i) *if* $t = 2p - 1$ *then*

$$\sigma(H^{2p}) \cap R^* = \emptyset, \tag{4.17}$$

where $R^* = R^0 \cup R^1$ (*see* (3.53));

(ii) *if* $t = 2m$, *then*

$$\sigma(H^{2m+1}) \cap R_* = \emptyset, \tag{4.18}$$

where $R_* = R^+ \cup R^-$ (*see* (3.54)).

Proof. By Lemmas 3 and 4

$$R^0 \cap R^1 = \emptyset \quad \text{and} \quad R^+ \cap R^- = \emptyset,$$

so we need to explain that *four* sets

$$R^0 \cap \sigma(h^*), \quad R^1 \cap \sigma(h^*), \quad R^+ \cap \sigma(h_*), \quad R^- \cap \sigma(h_*) \tag{4.19}$$

(where $h^* = H^{2p}$ in (i) and $h_* = H^{2m+1}$ in (ii)) are empty. The analysis of these four cases is almost identical. Let us give all details to prove (ii)-subcase

$$R^+ \cap \sigma(h_*) = 0. \tag{4.20}$$

If (4.20) does not hold, then for some $\mu \in \sigma(h_*)$

$$\delta^+(\mu) \equiv \delta^+(\mu; \alpha) = 0. \tag{4.21}$$

By (3.46) it implies that $\exists a^+ \neq 0,\ a^+ \in \mathbb{C}^m$ such that (see (3.44)–(3.46))

$$(H_m^+ - \mu)a^+ = 0, \quad a^+ = (A_j^+)_1^{2m-1}, \quad j \text{ odd}. \tag{4.22}$$

Notice that $A_{2m-1}^+ \neq 0$; otherwise by

$$q_{2m-1}A_{2m-3}^+ + (d_{2m-1} - \mu)A_{2m-1}^+ = 0 \tag{4.23}$$

we had $A_{2m-3}^+ = 0$ as well, and a backward induction by lines of (4.22) shows that $a^+ = 0$. But it is NOT the case.

Of course, in (4.22) H_m^+ is a submatrix of $H^+ \in (3.44)$, and

$$d_k = k^2, \quad q_k = 2\alpha(2m - 1 + k), \quad p_k = 2\alpha(2m - 1 - k). \tag{4.24}$$

With $\mu \in \sigma(h_*),\ h_* = H^{2m+1}$, we have an eigenvector $c \neq 0$,

$$(h_* - \mu)c = 0. \tag{4.25}$$

By Lemma 8 μ has a (geometric) multiplicity 1, and $Y \equiv \ell^2(F)$, where F is the set of all odd integers $k \geqslant 2m + 1$ can be decomposed as a direct sum (not necessarily orthogonal)

$$Y = Im\, P + Im\,(1 - P), \tag{4.26}$$

with

$$P = \frac{1}{2\pi i} \int_{|\mu - z| = \varepsilon} (z - h_*)^{-1} dz, \tag{4.27}$$

where

$$\varepsilon = \frac{1}{2} \min\{|\mu - \xi| : \ \xi \in \sigma(h_*),\ \xi \neq \mu\}.$$

Now we will use the h_*'s properties; it is a restriction of K^{even}, or K, on its invariant subspace Y. The operator $K = K_{Per^-}$ is similar to a self-adjoint operator L_{Per^-}. [This is not the case if a, b in (2.1) and (2.4) are not real; see further comment in Section 5.5.] Therefore, the geometric multiplicity of each h_*-eigenvalue is equal to its algebraic multiplicity. Lemma 8 implies that

$$\dim\ Im\ P = 1, \quad \text{and} \quad Im\ P = \{\xi c : \ \xi \in \mathbb{C}\}. \tag{4.28}$$

Put $U = Im\,(1 - P)$; then (4.26) can be written as

$$Y = \{\xi c\} + U, \quad h_* U \subset U, \tag{4.29}$$

$$\sigma(h_*|U) = \sigma(h_*) \setminus \{\mu\}. \tag{4.30}$$

Of course, $\begin{pmatrix} 0 \\ c \end{pmatrix}$ is a μ-eigenvector of K^{even} [see (3.43)–(3.51)]. Let us try to find another μ-eigenvector of the form $\begin{pmatrix} a^+ \\ y \end{pmatrix}$, where $a^+ \in$ (4.22), $y \in Y$ or even $y \in U$.

We have

$$(K^{\text{even}} - \mu) \begin{pmatrix} a^+ \\ y \end{pmatrix} = \begin{bmatrix} (H_m^+ - \mu) a^+ \\ \tau A_{2m-1}^+ e_{2m+1} + (H^{2m+1} - \mu) y \end{bmatrix}, \tag{4.31}$$

where $\tau = q_{2m+1} = 2\alpha \cdot 4m$. By (4.29)

$$e_{2m+1} = \gamma c + u, \qquad \gamma \in \mathbb{C}, \ u \in U. \tag{4.32}$$

Choose $y = y^* \in U$ in such a way that

$$\tau A_{2m-1}^+ u + (H^{2m+1} - \mu) y^* = 0. \tag{4.33}$$

By (4.30) the operator $(h_* - \mu)|U$ is invertible, so

$$y^* = (\mu - h_*)^{-1} \tau A_{2m-1}^+ u \tag{4.34}$$

is well defined; it solves Eq. (4.33). Therefore, by (4.31),

$$(K^{\text{even}} - \mu) \begin{pmatrix} a^+ \\ y^* \end{pmatrix} = \begin{bmatrix} 0 \\ \tau A_{2m-1}^+ \gamma c \end{bmatrix}, \tau = 8\alpha m \neq 0. \tag{4.35}$$

We have no control on γ; it comes from (4.32). Let us analyze the alternative: $\gamma = 0$ or $\gamma \neq 0$.

If $\gamma = 0$, with $a^+ \neq 0$, we have two linearly independent μ-eigenvectors $\begin{pmatrix} 0 \\ c \end{pmatrix}$ and $\begin{pmatrix} a^+ \\ y^* \end{pmatrix}$ for K^{even}. But it is impossible, as we noticed in Section 2, (2.14)–(2.21).

If $\gamma \neq 0$ then the coefficient $\tilde{\gamma} = \tau A_{2m-1}^+ \gamma$ in (4.35) is not zero as well by (4.31) and (4.23). In this case $f_0 = \begin{pmatrix} 0 \\ c \end{pmatrix}$ and $f_1 = \begin{pmatrix} a^+ \\ y^* \end{pmatrix}$ give us a Jordan block because

$$(K^{\text{even}} - \mu) f_0 = 0 \quad \text{and} \quad (K^{\text{even}} - \mu) f_1 = \tilde{\gamma} f_0, \quad \tilde{\gamma} \neq 0. \tag{4.36}$$

But, this is impossible because the operator $K = K^{\text{even}} + K^{\text{odd}}$ is similar to a self-adjoint operator L, and its invariant subspace $E = \textit{span}\,\{f_0, f_1\}$ should have *TWO* linearly independent μ-eigenvectors. This contradiction completes the proof of the claim in (4.20). As we noticed, other three sets in (4.19) could be analyzed in the same way to prove that they are empty. □

3. In Lemmas 5, 6 we showed that any eigenvalue μ of multiplicity 1

(i) for K_{Per^+} when $t = 2p - 1$ is a root of δ^0 or δ^1 (see (3.23)–(3.25));
(ii) for K_{Per^-} when $t = 2m$ is a root of δ^+ or δ^- (see (3.46)–(3.52)).

Now we will prove that the inverse is true.

Lemma 10. *Let α be real and nonzero.*

(i) *If $t = 2p - 1$, then each $\mu \in R^*$ is simple root of δ^0 or δ^1, and μ is an eigenvalue of K_{Per^+} of multiplicity* 1.
(ii) *If $t = 2m$, then each $\mu \in R_*$ is simple root of δ^+ or δ^-, and μ is an eigenvalue of K_{Per^-} of multiplicity* 1.

Proof. Again we have four cases: δ^0 or δ^1 in (i), and δ^+ or δ^- in (ii).The analysis of these four cases is almost identical. Let us give all the details in the (i)-subcase δ^1.

Assume that

$$\delta^1(\mu) = 0. \tag{4.37}$$

By Lemma 9 the operator $(h^* - \mu)$ is invertible. For brevity, let us write $g = H^2_{2p-2}$ (see (3.22.1), (3.25), (3.37)). If μ as a root of $\delta^1(z) = \det(z - g)$ has multiplicity $\geqslant 2$, then there are two linearly independent vectors

$$b_1^+,\ b_2^+ \in \mathbb{C}^{p-1}, \quad b_\sigma^+ = \{B_\sigma^+(j)\}_2^{2p-2}, \quad j \text{ even}, \quad \sigma = 1, 2, \tag{4.38}$$

such that

$$(g - \mu)b_1^+ = 0 \quad \text{and} \quad (g - \mu)b_2^+ = \xi b_1^+. \tag{4.39}$$

Put

$$y_1 = (\mu - h^*)^{-1}\tau B_1^+(2p - 2)e_{2p} \tag{4.40}$$

and

$$y_2 = (\mu - h^*)^{-1}\left[-\xi y_1 + \tau B_2^+(2p - 2)e_{2p}\right]. \tag{4.41}$$

These vectors are well defined because by Lemma 9(i) the operator $(\mu - h^*)$ is invertible.

Then [compare (4.31)–(4.35)] by (4.39)–(4.41)

$$\left(K^{\text{odd}} - \mu\right)\begin{bmatrix} b_1^+ \\ y_1 \end{bmatrix} = \begin{bmatrix} (g-\mu)b_1^+ \\ \tau B_1^+(2p-2)e_{2p} + (h^* - \mu)^{-1}y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\left(K^{\text{odd}} - \mu\right)\begin{bmatrix} b_2^+ \\ y_2 \end{bmatrix} = \begin{bmatrix} (g-\mu)B_2^+ \\ \xi y_1 \end{bmatrix} = \xi\begin{bmatrix} b_1^+ \\ y_1 \end{bmatrix},$$

or with $f_\sigma = \begin{bmatrix} b_\sigma^+ \\ y_\sigma \end{bmatrix}, \ \sigma = 1, 2,$

$$\left(K^{\text{odd}} - \mu\right) f_1 = 0, \qquad \left(K^{\text{odd}} - \mu\right) f_2 = \xi f_1. \tag{4.42}$$

By (4.38) f_1 and f_2 are linearly independent odd functions. Again [compare the end of the proof of Lemma 9, after (4.34)] if $\xi = 0$, then we have TWO linearly independent odd μ-eigenfunctions for K that is impossible. If $\xi \neq 0$ then f_1 and f_2 give us a Jordan block by (4.42), but it is impossible either, because K is similar to the self-adjoint operator L. It proves that μ is a δ^1-root of multiplicity 1. In this case a vector $b_1^+, \ b_1^+ \in (4.39)$, does exist, and with $y_1 \in (4.40)$ give an odd μ-eigenfunction

$$f_1 = \begin{bmatrix} b_1^+ \\ y_1 \end{bmatrix} \neq 0 \tag{4.43}$$

for K or K^{odd}. If μ is of multiplicity $\geqslant 2$ for K then there exist an even function (vector)

$$A = (a, a') \neq 0 \tag{4.44}$$

(see (3.19), (3.21)) such that (3.21.1)–(3.21.2) hold. If $a \neq 0$, then by (3.21.1),

$$\delta^0(\mu) = 0;$$

however, by Lemma 1, (4.37) implies that $\delta^0(\mu) \neq 0$. With $a = 0$, (4.44) requires $a' \neq 0$. But then by (3.21.2)

$$\left(H^{2p} - \mu\right) a' = 0 \quad \text{for} \quad \mu \in \sigma(H^{2p}) \tag{4.45}$$

which contradicts Lemma 9, (4.17). Therefore, $\mu \in (4.37)$ is a simple eigenvalue of K. Lemma 10 is proven. □

4. The technical lemmas in this section have quite elementary proofs; sometimes—and it is often essential—these proofs use the fact that our non-symmetric matrices represent operators similar to self-adjoint ones.

Direct analysis of these matrices and polynomials δ^0, δ^1, $\delta^\pm$ and their zeroes can be done with a help of few basic facts about OPS, *orthogonal polynomial sequences*. Let us remind these facts (we refer to [6] for details and proofs; see Sections 1.4–1.6, pp. 18–28).

For any sequences $\{c_n\}_1^\infty$ of reals and $\{\lambda_n\}_1^\infty, \ \lambda_n \neq 0$, let us define polynomials

$$P_n(x) = (x - c_n) P_{n-1} - \lambda_n P_{n-2}(x), \quad n = 1, 2, \ldots, \tag{4.46}$$

$$P_{-1}(x) \equiv 0, \quad P_0(x) \equiv 1 \tag{4.47}$$

(compare (4.1) and (4.6), [6, pp. 18–21]). Then for each $n \in \mathbb{N}$ the zeroes of $P_n(x)$ are real and simple [6, Theorem 5.2, p. 27]. Let us denote its zeroes by $x^n(i)$ being ordered by increasing size, i.e.,

$$x^n(1) < x^n(2) < \cdots < x^n(i) < x^n(i+1) < \cdots < x^n(n). \tag{4.48}$$

The zeroes of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other, i.e.,

$$x^{n+1}(i) < x^n(i) < x^{n+1}(i+1) < \cdots < x^n(i+1), \quad i = 1, \ldots, n \tag{4.49}$$

[6, Theorem 5.3, p. 28].

These statements are useful to us because δ^0 and δ^1 could be considered as two consequent terms of such OPS. Indeed, with $t = 2p - 1$ the matrix H^0_{2p-2} in (3.24), (3.25) and (3.17) is

$$\begin{bmatrix} 0 & 2\alpha \cdot 2(p-1) & & & \\ 4\alpha \cdot 2p & 2^2 & 2\alpha \cdot 2(p-2) & & \\ 0 & 2\alpha \cdot 2(p+1) & 4^2 & 2\alpha \cdot 2(p-3) & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \\ & & & 2\alpha \cdot 2 \cdot 2(p-1) & (2p-2)^2 \end{bmatrix}. \tag{4.50}$$

All elements on the off-diagonals are not zeros. We go backward; put

$$Q_1(x) = (2(p-2))^2 - x, \tag{4.51}$$

$$Q_k(x) = \det\left[H^{2(p-k)}_{2p-2} - x\right]. \tag{4.52}$$

As we already noticed

$$Q_{k+1}(x) = (c_{k+1} - x)Q_k(x) - \lambda_{k+1}Q_{k-1}(x), \tag{4.53}$$

where

$$c_k = (2(p-k))^2, \quad 1 \leqslant k \leqslant p, \tag{4.54}$$

$$\lambda_k = (k-1)(2p-k)16\alpha^2, \quad 2 \leqslant k \leqslant p-1, \tag{4.55}$$

$$\lambda_p = 32\alpha^2(p-1)p. \tag{4.56}$$

We can (arbitrarily) put

$$c_k = 0, \quad \lambda_k = 1 \quad \text{for } k > p, \tag{4.57}$$

to have OPS well-defined for all $n \in \mathbb{N}$, but we are really interested only in two polynomials

$$\delta^0(x) \equiv Q_p(x) \quad \text{and} \quad \delta^1(x) \equiv Q_{p-1}(x). \tag{4.58}$$

If $x^0(i),\ 0 \leqslant i \leqslant p-1$, and $x^1(i),\ 1 \leqslant i \leqslant p-1$, are the zeros of δ^0 and δ^1 being ordered by increasing size as (4.48), by (4.49) we have

$$x^0(0) < x^1(1) < x^0(1) < \cdots < x^1(i) < x^0(i) < \cdots < x^0(p-1). \tag{4.59}$$

Therefore, the roots of δ^0 and δ^1 are real and distinct [we knew this by Lemma 3], and they interlace, i.e., (4.59) holds for all $\alpha \neq 0$. The latter is an important corollary of (4.46)–(4.49).

Analysis of zeros of δ^+ and δ^- is a little more complicated. Recall that (3.46) defines these polynomials (with parameter α) by matrices (3.44)

$$H_m^{\pm} = \begin{bmatrix} 1 \pm 4\alpha m & 4\alpha(m-1) & & & \\ 4\alpha(m+1) & 3^2 & 4\alpha(m-2) & & \\ 0 & 4\alpha(m+2) & 5^2 & 4\alpha(m-3) & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \\ & & & 4\alpha(2m-1) & (2m-1)^2 \end{bmatrix}. \tag{4.60}$$

Now δ^+ and δ^-,

$$\delta^{\pm} = \det\left(H_m^{\pm} - \mu\right) \tag{4.61}$$

are polynomials of the same order m but OPS theory helps us if we notice (compare with Lemma 4) the following. The left column is a sum of

$$\begin{bmatrix} 1 \\ 4\alpha(m+1) \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pm 4\alpha m \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}. \tag{4.62}$$

This decomposition implies that

$$\delta^{\pm}(x;\alpha) = P(x;\alpha) \pm 4\alpha m Q(x;\alpha), \tag{4.63}$$

where P and Q are consequent polynomials of OPS we could construct by using the matrix

$$\begin{bmatrix} 1 & 4\alpha(m-1) & & & \\ 4\alpha(m+1) & 3^2 & 4\alpha(m-2) & & \\ 0 & 4\alpha(m+2) & 5^2 & 4\alpha(m-3) & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \\ & & & 4\alpha(2m-1) & (2m-1)^2 \end{bmatrix} \tag{4.64}$$

for a backward procedure in the same way as we used matrix (4.50) to explain that δ^0 and δ^1 in (4.58) have this property. Let

$$z_j(\alpha),\ 1 \leqslant j \leqslant m \quad \text{and} \quad \tilde{z}_j(\alpha),\ 2 \leqslant j \leqslant m, \tag{4.65}$$

be the zeros of P and Q in (4.63). Again by (4.49) they interlace so

$$z_1(\alpha) < \tilde{z}_2(\alpha) < z_2(\alpha) < \cdots < \tilde{z}_m(\alpha) < z_m(\alpha). \tag{4.66}$$

But these zeros are not (case (4.59) was easy) the zeros of our polynomials $\delta^{\pm}$ in (4.63). Still (4.66) is important and useful. Let

$$\xi_j^{\pm}(\alpha), \quad 1 \leqslant j \leqslant m, \tag{4.67}$$

be zeros of $\delta^{\pm}$. We know that

$$P(z;0) = (1-z)Q(z;0) = (1-z)\prod_{j=2}^{m}\left[(2j-1)^2 - z\right] \tag{4.68}$$

and

$$\xi_j^{\pm}(0) = (2j-1)^2, \quad 1 \leqslant j \leqslant m, \tag{4.69}$$

$$z_j(0) = (2j-1)^2, \quad 1 \leqslant j \leqslant m, \quad \tilde{z}_j(0) = (2j-1)^2, \quad 2 \leqslant j \leqslant m. \tag{4.70}$$

We know by the above analysis that $z_j(\alpha)$, $1 \leqslant j \leqslant m$, are distinct for all real α, and $\tilde{z}_j(\alpha)$, $2 \leqslant j \leqslant m$ are distinct as well. Therefore they are analytic functions of $\alpha \in \mathbb{R}$ as roots of polynomials with higher coefficient ± 1. Eq. (4.69) tells us that these roots are distinct if $\alpha = 0$ so they remain distinct for small enough α, certainly, if $|\alpha| < 1/7$. Let us assume for a while that $|\alpha| < 1/7$. We want to show that for $0 < \alpha < 1/7$

$$\xi_1^-(\alpha) < \xi_1^+(\alpha) < \xi_2^-(\alpha) < \cdots < \xi_m^-(\alpha) < \xi_m^+(\alpha). \tag{4.71}$$

Because P and Q are of order m and $m-1$, the root $z_1(\alpha)$ is special, so first we prove that

$$\xi_1^-(\alpha) < \xi_1^+(\alpha), \quad 0 < \alpha < 1/7. \tag{4.72}$$

With notations (4.65) and (4.67)

$$P(z,\alpha) = \prod_1^m (z_k(\alpha - z) = (z_1(\alpha) - z)R_1(z,\alpha), \tag{4.73}$$

where

$$R_1 = \prod_2^m (z_k(\alpha) - z) \tag{4.74}$$

and

$$Q(z,\alpha) = \prod_2^m (\tilde{z}_k(\alpha - z)) \equiv \tilde{R}_1(z;\alpha). \tag{4.75}$$

Then

$$P(\xi_1^+(\alpha);\alpha) = (z_1(\alpha) - \xi_1^+(\alpha))R_1^{\pm}(\alpha), \tag{4.76}$$

where

$$R_1^{\pm}(\alpha) = R_1(\xi_1^{\pm}(\alpha);\alpha) = \prod_2^m (z_k(\alpha) - \xi_1^{\pm}(\alpha)) \tag{4.77}$$

and

$$Q(\xi_1^{\pm}(\alpha);\alpha) = \tilde{R}_1^{\pm}(\alpha), \tag{4.78}$$

where

$$\tilde{R}_1^{\pm}(\alpha) = \prod_2^m (\tilde{z}_k(\alpha) - \xi_1^{\pm}(\alpha)). \tag{4.79}$$

All these functions are analytic on α for $|\alpha| < 1/7$. Our basic equation for $\xi_1^{\pm}$ is (4.63); it implies

$$(z_1(\alpha^0 - \xi_1^{\pm}(\alpha))R_1^{\pm}(\alpha) \pm 4m\alpha \tilde{R}_1^{\pm}(\alpha) = 0, \tag{4.80}$$

$$\xi_1^{\pm}(\alpha) = z_1(\alpha) \pm 4m\alpha \left(\tilde{R}_1^{\pm}(\alpha)/R_1^{\pm}(\alpha)\right). \tag{4.81}$$

By (4.77), (4.79) and (4.69)

$$R_1^{\pm}(0) = \tilde{R}_1^{\pm} = \prod_2^m = \left[(2j-1)^2 - 1\right]. \tag{4.82}$$

Therefore, for some $\alpha_m^* > 0$ and $-\alpha_m^* < \alpha < \alpha_m^*$ the ratios $\tilde{R}_1^+/R_1^+$ and $\tilde{R}_1^-/R_1^-$ on the right-hand side of (4.81) are certainly positive and between 1/2 and 2, so

$$\xi_1^- < z_1(\alpha) < \xi_1^+(\alpha), \quad 0 < \alpha < \alpha_m^* \tag{4.83}$$

and

$$\xi_1^+ < z_1(\alpha) < \xi_1^-(\alpha), \quad -\alpha_m^* < \alpha < 0. \tag{4.84}$$

Now we consider the roots $\xi_j^{\pm},\ 2 \leqslant j \leqslant m$. For $2 \leqslant k \leqslant m$, as in (4.73)–(4.79)

$$P(z,\alpha) = (z_1(\alpha) - z)R_k(z;\alpha), \tag{4.85}$$

where

$$R_k(z,\alpha) = \prod_{\substack{j=2 \\ j \neq k}}^m (z_j(\alpha) - z) \tag{4.86}$$

and

$$Q(z,\alpha) = (\tilde{z}_k(\alpha) - z) \prod_{\substack{j=2 \\ j \neq k}}^m (\tilde{z}_j(\alpha) - z) \equiv (\tilde{z}_k(\alpha) - z)\tilde{R}_k(z;\alpha). \tag{4.87}$$

Put

$$R_k^{\pm}(\alpha) = P_k(\xi_k^{\pm}(\alpha);\alpha) \tag{4.88}$$

and

$$\tilde{R}_k^{\pm}(\alpha) = \tilde{P}_k(\xi_k^{\pm}(\alpha);\alpha). \tag{4.89}$$

As in (4.82)

$$R_k^{\pm}(0) = \tilde{R}_k^{\pm}(0) = \prod_{\substack{j=2 \\ j \neq k}}^{m} \left[(2j-1)^2 - 1 \right]. \tag{4.90}$$

All these functions are analytic on α for $|\alpha| < 1/7$, and for some $\alpha^{**} > 0$ (the same for all $k,\ 2 \leqslant k \leqslant m$) if α is real and $|\alpha| < \alpha^{**}$, then we have

$$1/2 < \tilde{R}_k^{+}(\alpha)/R_k^{+}(\alpha), \quad \tilde{R}_k^{-}(\alpha)/R_k^{-}(\alpha) < 2. \tag{4.91}$$

The basic equation (4.63) for $\xi_k^{\pm}(\alpha)$ implies:

$$(z_1(\alpha) - \xi_k^{\pm}(\alpha))(z_k(\alpha) - \xi_k^{\pm}(\alpha)) R_k^{\pm}(\alpha) \pm 4m\alpha(\tilde{z}_k(\alpha) - \xi_k^{\pm}(\alpha))\tilde{R}_k^{\pm}(\alpha) = 0 \tag{4.92}$$

and

$$z_k(\alpha) - \xi_k^{\pm}(\alpha) \pm 4m\alpha \frac{\tilde{R}_k^{\pm}(\alpha)}{R_k^{\pm}(\alpha)} \cdot \frac{\tilde{z}_k(\alpha) - z_k(\alpha) + z_k(\alpha) - \xi_k^{\pm}(\alpha)}{z_1(\alpha) - \xi_k^{\pm}(\alpha)} = 0, \tag{4.93}$$

or

$$\xi_k^{\pm}(\alpha) = z_k(\alpha) \pm 4m\alpha(\tilde{z}_k(\alpha) - z_k(\alpha)) S_k^{\pm}(\alpha), \tag{4.94}$$

where

$$S_k^{\pm}(\alpha) = \frac{\tilde{R}_k^{\pm}}{R_k^{\pm}} \cdot \frac{1}{z_1(\alpha) - \xi_k^{\pm}(\alpha)} \left[1 \pm 4m\alpha \frac{\tilde{R}_k^{\pm}}{R_k^{\pm}} \cdot \frac{1}{z_1(\alpha) - \xi_k^{\pm}(\alpha)} \right]^{-1} \tag{4.95}$$

with

$$\xi_k^{\pm}(0) = (2k-1)^2 \quad \text{and} \quad z_1(0) = 1. \tag{4.96}$$

For $|\alpha| < \alpha_m^{**}$ the denominator

$$z_1(\alpha) - \xi_k^{\pm}(\alpha) < \left(1 - (2k-1)^2 \right) + 1 \leqslant -7 \quad \text{if} \ \ k \geqslant 2, \tag{4.97}$$

is negative and

$$S_k^{\pm}(\alpha) < 0, \quad |\alpha| \leqslant \alpha_m^{**}. \tag{4.98}$$

By interlacing (4.66) we obtain

$$0 < z_k(\alpha) - \tilde{z}_k(\alpha), \tag{4.99}$$

so (4.94), (4.98) and (4.99) imply for $0 < \alpha < \alpha_m^{**}$ that

$$\xi_k^{-}(\alpha) < z_k(\alpha) < \xi_k^{+}(\alpha) \tag{4.100}$$

and for $-\alpha < \alpha < 0$

$$\xi_k^{+}(\alpha) < z_k(\alpha) < \xi_k^{-}(\alpha), \quad 2 \leqslant k \leqslant m. \tag{4.101}$$

For $k = 1$ it is proven in (4.83) and (4.84).

We explained (see Lemma 4) that

$$R^+ \cap R^- = \emptyset \quad \text{for} \ \alpha \neq 0. \tag{4.102}$$

Therefore, the interlacing

$$\xi_1^-(\alpha) < \xi_1^+(\alpha) < \xi_2^-(\alpha) < \cdots < \xi_m^-(\alpha) < \xi_m^+(\alpha), \tag{4.103}$$

which we have just proven for $0 < \alpha < \alpha_m^{**}$ will remain valid for all $\alpha > 0$. The same extension by continuation will preserve the interlacing

$$\xi_1^+(\alpha) < \xi_1^-(\alpha) < \xi_2^+(\alpha) < \cdots < \xi_m^+(\alpha) < \xi_m^-(\alpha) \tag{4.104}$$

for all $\alpha < 0$.

It is interesting to notice for the roots of $\delta^\pm$ that their ordering changes (see (4.103) and (4.104)) when α goes from positive to negative. (It does not happen in the Per^+ case (see (4.59)). But this is not surprising because

$$\delta^0(\mu; \alpha) = \delta^0(\mu; -\alpha) \quad \text{and} \quad \delta^1(\mu; \alpha) = \delta^1(\mu; -\alpha), \tag{4.105}$$

i.e., δ^0 and δ^1 are even with respect to α, but $\delta^+(\mu; -\alpha) = \delta^-(\mu; \alpha)$.

5. We can summarize the analysis and results of this section as the following.

Theorem 11. *Let*

$$v(x) = a\cos 2x + b\cos 4x, \quad a = -4\alpha t, \ b = -2\alpha^2 \quad real, \ \alpha \neq 0, \tag{4.106}$$

be a potential of the Hill operator

$$Ly = -y'' + v(x)y, \quad 0 \leqslant x \leqslant \pi. \tag{4.107}$$

(i) *If $t = 2p - 1$, $p \geqslant 1$, and $bc = Per^+$ then the first $2p - 1$ eigenvalues are simple, and others are double,*

$$\lambda_0^+(\alpha) < \lambda_2^-(\alpha) < \lambda_2^+(\alpha) < \cdots < \lambda_{2(p-1)}^-(\alpha) < \lambda_{2(p-1)}^+(\alpha)$$
$$< \lambda_{2p}^-(\alpha) < \lambda_{2p}^+(\alpha) < \lambda_{2j}^-(\alpha) = \lambda_{2j}^+(\alpha) \qquad j > p. \tag{4.108}$$

Moreover, the eigenvalues $\lambda_{2k}^+(\alpha)$, $0 \leqslant k \leqslant p - 1$, are zeros of the polynomial $\delta^0(\mu, \alpha)$, and the eigenvalues $\lambda_{2k}^-(\alpha)$, $0 \leqslant k \leqslant p - 1$, are zeros of the polynomial $\delta^1(\mu, \alpha)$.

(ii) *If $t = 2m$, $m \geqslant 1$, and $bc = Per^-$, then the first $2m$ eigenvalues are simple and others are double, i.e.,*

$$\lambda_1^\pm(\alpha) < \lambda_3^\pm(\alpha) < \cdots \lambda_{2m-1}^\pm(\alpha) < \lambda_{2m+1}^\pm(\alpha) < \cdots \tag{4.109}$$

and

$$\lambda_{2j-1}^-(\alpha) < \lambda_{2j-1}^+(\alpha), \ \ 1 \leqslant j \leqslant m, \quad \lambda_{2j+1}^-(\alpha) = \lambda_{2j+1}^-(\alpha), \ \ j \geqslant m. \tag{4.110}$$

Moreover, the eigenvalues $\lambda_{2j-1}^+(\alpha)$, $1 \leqslant j \leqslant m$, are zeros of the polynomial $\delta^+(\mu, \alpha)$ if $\alpha > 0$, and of the polynomial $\delta^-(\mu, \alpha)$ if $\alpha < 0$, and v.v., the eigenvalues $\lambda_{2j-1}^-(\alpha)$,

$1 \leqslant j \leqslant m$, are zeros of the polynomial $\delta^-(\mu, \alpha)$ if $\alpha > 0$, and of the polynomial $\delta^+(\mu, \alpha)$ if $\alpha < 0$.

6. Just to demonstrate how the structure of spectra changes when the parameters a, b cross the integer levels of t in (3.2) we consider pockets of instability of one-parametric family of potentials

$$v(x) = -\tau(8\cos 2x + 8\cos 4x). \tag{4.111}$$

According to (3.2)

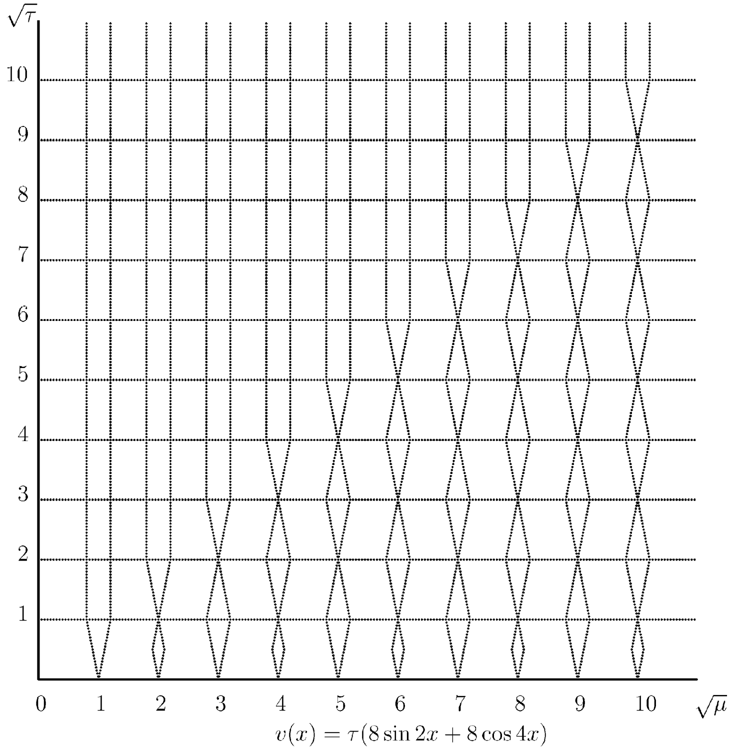
$$8(-8\tau)t^2 + (8\tau)^2 = 0, \tag{4.112}$$

so

$$t = \tau^2. \tag{4.113}$$

Therefore, all eigenvalues in the case of potential $v \in$ (4.111) are simple (the zones of instability are open) if τ^2 is not an integer.

If $t = \tau^2$ is an integer then according to Theorem 1 the first t zones are open, the $(t+1)$st zone is closed, and then they interlace, i.e., the zones $t + 2m$, $m = 1, 2, \ldots,$ are open and the zones $t + 2p - 1$, $p = 1, 2, \ldots,$ are closed. It is shown in the following diagram.



$v(x) = \tau(8\sin 2x + 8\cos 4x)$

We need to point out that this is a diagram, not a real graph. It ignores the values of $\lambda^{\pm}$ and how two curves $\lambda_n^-(\tau)$, $\lambda_n^+(\tau)$ intersect at the integer τ^2. Even at $\tau = 0$ the diagram does not show the level of contact of these curves with the same tangent (vertical) line.

5. Comments; conclusion

1. The crucial step in killing a higher-frequency term of potential (2.4) is transformation (2.7) used by Ince [21] in the 1920s, and Magnus and Winkler in the 1950s. Of course, in the 1980s such type of gauge transform became routine in both mathematical and physical literature, but it was not a standard procedure in the 1930s or even in the 1950s. True, one can find “Sommerfeld procedure” as Razavy [33] put it, in the 1929 book [36], and occasionally in the 1930s and 1940s. But even the Razavy’s observation [33] in 1980 that the bistable potential in the Schrödinger operator

$$L\psi = \psi'' + \left(\varepsilon + \frac{1}{8}\xi^2 + (n-1)\xi \cosh 2x - \frac{1}{8}\cosh 4x\right)$$

following the Sommerfeld procedure

$$\psi = \exp\left(-\frac{1}{4}\cosh 2x\right)\varphi(x)$$

brings us to an operator $K = E^{-1}LE$,

$$K\varphi = \varphi'' - \xi \sinh 2x\varphi' + (\varepsilon + n\xi \cosh 2x)\varphi$$

without terms of the rate 4, has been considered as a breaking news. Of course, this is the same transform (2.5)–(2.11) used by Ince in 1923 if you change x to ix.

Klotter and Kotowski in 1943 did numerical calculations [23] to see the behavior of the eigenvalues of this operator but they used the five-diagonal matrix to present operator (2.6) in trigonometric basis as it directly follows from (2.4). Multiplication by this potential is, in an obvious way, a five diagonal matrix.

2. A tridiagonal matrix representation led Magnus and Winkler [48] to Theorem 7.9 in [26, p. 107], because *a zero* on the off-diagonal changes drastically the spectra and gives a very special finite-dimensional subspace (invariant for K or L, or for adjoint K^*). It makes the work of Magnus and Winkler in the 1950s quite a remarkable piece—if we follow the language of the 1990s [39,40,14]—in the theory of quasi-exactly solvable differential equations, or QES. Indeed, this is one of the canonical examples in this QES-theory (see (60) and (65) in Turbiner [38]). But one cannot see in this literature any mentioning of Ince [20–22] or Magnus and Winkler results from the 1950s [48], or their exposition in the books [1,26] published in the 1960s.

3. Our Theorem 11 sharpens the results of Magnus and Winkler by giving complete analysis of spectra of a “head” matrix (or, the algebraic sector, as Shifman and Turbiner say in [34]) and a “tail” matrix and their relationship. By (not well motivated) analogy we can

ask whether the same spectral properties are observed in quasi-exactly solvable equations of one variable (see their catalogue in [38] or [39,40]).

A. Are all eigenvalues in the algebraic sector simple?

Of course, the answer is positive, if one can bring this block (by some gauge transformation?) to tridiagonal matrix without zeroes on the off-diagonals. In our context Lemma 10, together with Lemma 9, gives a positive answer to Question A.

Next two questions are vague because, with great emphasis on an algebraic sector (finite-dimensional invariant subspace), QES-theory does not define in a canonical way a remainder, or a compliment, or a "tail" block of the differential operator L which is quasi-exactly solvable.

B. Are the eigenvalues of such an operator L which is determined by the tail, or which do not come from the algebraic sector, double, i.e., do they have multiplicity 2?

In our context the answer is YES because the "tail" operators in subspaces of even and odd functions are just identical; see (3.21.2) and (3.22.2) in Per^+-case, and (3.50)–(3.51) in Per^--case.

Of course, if A and B have positive answers, then the eigenvalues of these two classes could not coincide. [See Lemmas 9 and 10 in our context.] But we do not know this yet, so let us ask the following question.

C. Is it true that eigenvalues from the algebraic sector could not coincide with eigenvalues coming from outside the algebraic sector?

4. Maybe, in these questions of Section 5.3 we implicitly assume that the operator L under the consideration is selfadjoint and parameters are real. Certainly, it was the case in our analysis of operator (1.2) with potential (2.1), or (2.23) + (2.55). But it is interesting to check which statements (from Proposition 1 to Lemma 10) and their proofs depend on the assumption that α is real. To be certain, let us now talk about positive $t > 0$ and complex α with $a = -4\alpha t$ and $b = -2\alpha^2$.

What Propositions 1 and 2 really showed is that for any $\alpha \in \mathbb{C} \setminus \{0\}$ the equation

$$- y'' - (4\alpha t \cos 2x + 2\alpha^2 \cos 4x) y = \lambda y \tag{5.1}$$

cannot have non-zero *even* and *odd* Per^+-solutions (if t is not odd) at the same time, and there could not be *even* and *odd* Per^--solutions (if t is not even).

Technical Lemmas 3 (and 4) and 8 hold for any matrices with complex entries as well.

In Lemmas 5 and 6 we have essentially the same effect as in the proofs of Proposition 1 and 2. It becomes more obvious if we point out that "multiplicity 1" there means a weaker assumption on "geometric multiplicity 1". The distinction is lost of course, if L is self-adjoint (and K is similar to L). So Lemmas 5 and 6 hold for any $\alpha \in \mathbb{C} \setminus \{0\}$ as well.

But in the proofs of Lemmas 9 and 10, as we have noticed there [2] we used in a critical way that K is similar to a self-adjoint operator L. The same should be said about the claim (a part of Theorem 11) that the roots of a polynomial $\delta^0(x; \alpha)$ are *simple*, i.e., the eigenvalues of the "head" (or of the algebraic sector) have ALGEBRAIC multiplicity 1. This is not necessarily true if α is complex. Let us consider explicit examples.

Example 1. Per^--case; $t = 4$, or $m = 2$. By (4.60)

[2] Five lines after (4.27) or the paragraph after (4.42).

$$\delta^{\pm}(z;\alpha) = \det\begin{bmatrix} 1 \pm 8\alpha - z & 4\alpha \\ 12\alpha & 9 - z \end{bmatrix} = z^2 - 10z + 9 \pm 8\alpha(9 - z) - 48\alpha^2$$

and

$$\delta^{+} = z^2 - (10 + 8\alpha)z + 9 + 72\alpha - 48\alpha^2,$$

$$\delta^{-} = z^2 - (10 - 8\alpha)z + 9 - 72\alpha - 48\alpha^2.$$

Roots of δ^{+} are

$$5 + 4\alpha \pm 4(1 - 2\alpha + 4\alpha^2)^{1/2}$$

and for δ^{-}

$$5 - 4\alpha \pm 4(1 + 2\alpha + 4\alpha^2)^{1/2}.$$

These roots $6 \pm i\sqrt{3}$ are of multiplicity 2,

$$\text{if} \quad \alpha = (1 \pm i\sqrt{3})/4 \quad \text{for} \ \ \delta^{+}, \tag{5.2}$$

or

$$\text{if} \quad \alpha = (-1 \pm i\sqrt{3})/4 \quad \text{for} \ \ \delta^{-}. \tag{5.3}$$

The operators K^{even} and K^{odd} have Jordan blocks (in their "heads") if (5.2), or (5.3), hold.

Example 2. This example is more interesting and more complicated because now δ^0 is a polynomial of degree 3. We consider Per^{+} -case; $t = 5$, or $p = 3$. By (4.50)

$$\delta^0(z;\alpha) = \det\begin{bmatrix} -z & 8\alpha & 0 \\ 24\alpha & 4 - z & 4\alpha \\ 0 & 16\alpha & 16 - z \end{bmatrix} = -(z^3 - 20z^2 + 64(1 - 4\alpha^2) + 3.2^{10}\alpha^2).$$

It has a double root a in the case of *three* values of α^2, or six values of α :

$$\alpha = \pm i0.14796395, \quad a = 2.057664008;$$

$$\alpha = \pm(-.5537604 + i.5717989), \quad a = 4.4300839 + i4.674391484;$$

$$\alpha = \pm(.5537604 + i.5717989), \quad a = 4.4300839 - i4.674391484.$$

But these three values of a are $L_{Per^{+}}$-eigenvalues of geometric multiplicity 1 anyway.

For curiosity, let us notice that

$$\delta^1(z,\alpha) = \det\begin{bmatrix} 4 - z & 4\alpha \\ 16\alpha & 16 - z \end{bmatrix} = z^2 - 20z + 64 - 64\alpha^2.$$

Its roots are $10 \pm \sqrt{36 + 64\alpha^2}$, so δ^1 has a root $+10$ of multiplicity 2 if $\alpha = \pm 3i/4$. Again, $L_{Per^{+}}$, or its restriction K^{odd}, has a Jordan block.

5. Examples in the previous subsection show that in Lemmas 9, 10 and Theorem 11 the assumptions that a, b be real, or L be self-adjoint, are important. But let us follow

[3,35,19,4,5,37] and raise a general question about the structure of spectral Riemann surfaces related to these problems. Of course, it would be interesting to change both α and t in complex plane, i.e., to consider $(\alpha, t) \in \mathbb{C}^2$ but for a while, let us talk about fixed positive t. Define, for each $t > 0$, *four* surfaces

$$G_0(t) = \{(\mu, \alpha) : \ \exists x \in \ell^2(2\mathbb{N} - 2) \ \text{ such that } \ H^0(\alpha)x = \mu x\},$$

$$G_1(t) = \{(\mu, \alpha) : \ \exists x \in \ell^2(2\mathbb{N}) \ \text{ such that } \ H^2(\alpha)x = \mu x\},$$

$$G^+(t) = \{(\mu, \alpha) : \ \exists x \in \ell^2(2\mathbb{N} - 1) \ \text{ such that } \ H^+(\alpha)x = \mu x\},$$

$$G^-(t) = \{(\mu, \alpha) : \ \exists x \in \ell^2(2\mathbb{N} - 1) \ \text{ such that } \ H^-(\alpha)x = \mu x\},$$

where for each parity H^0, H^2 are defined by (3.16)–(3.17), and $H^\pm$ are defined by (3.44).

What is the structure of these surfaces?

In the case of anharmonic oscillator equation such a question has been raised and solved by Bender and Wu [3]; see also [35,37]. The case of Mathieu–Hill operators has a longer history (see [30,31,4,5,19,42,43,45]).

If t is an integer then as we have seen in our text [but this is really the Turbiner's observation [37] about *any* quasi-exactly-solvable differential operator], G_0 and G_1 are split into two surfaces if t is odd, while G^+ and G^- are split into two surfaces if t is even, one of them being algebraic. These surfaces are zero-surfaces of polynomials δ^0 and δ^1, or δ^+ and δ^- respectively. Examples 1 and 2 in Section 5.4 give some branching points (of order 2) of these surfaces.

But their structure in general remains a mystery.

Acknowledgments

The authors thank Professor Alexander Turbiner and Professor Hans Volkmer for detailed information about the literature on related matters and further discussions. The first author (P. D.) acknowledges the hospitality of The Ohio State University during the academic year 2003/2004. The second author (B. M.) is grateful to the Institute of Nuclear Sciences and the Institute of Applied Mathematics and System Analysis, UNAM, Mexico City, Mexico, whose hospitality he enjoyed in Spring 2004.

References

[1] F.M. Arscott, Periodic differential equations. An introduction to Mathieu, Lame, and allied functions, International Series of Monographs in Pure and Applied Mathematics, vol. 66, A Pergamon Press Book, The Macmillan Co., New York, 1964, p. x+284pp.
[2] J. Avron, B. Simon, The asymptotics of the gap in the Mathieu equation, Ann. Phys. 134 (1981) 76–84.
[3] C. Bender, T.T. Wu, An anharmonic oscillator, Phys. Rev. 184 (5) (1969) 1231–1260.
[4] G. Blanch, Numerical aspects of Mathieu's equations, Rend. Circ. Mat. Palermo 15 (1966) 51–97.
[5] G. Blanch, D.S. Clemm, The double points of Mathieu's differential equation, Math. Comp. 23 (105) (1969) 97–108.
[6] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[7] P. Djakov, B. Mityagin, Smoothness of Schrödinger operator potential in the case of Gevrey type asymptotics of the gaps, J. Funct. Anal. 195 (2002) 89–128.

[8] P. Djakov, B. Mityagin, Spectral gaps of the periodic Schrödinger operator when its potential is an entire function, Adv. Appl. Math. 31 (3) (2003) 562–596.

[9] P. Djakov, B. Mityagin, Spectral triangles of Schrödinger operators with complex potentials, Selecta Math. 9 (2003) 495–528.

[10] P. Djakov, B. Mityagin, The asymptotics of spectral gaps of 1D Dirac operator with cosine potential, Lett. Math. Phys. 65 (2003) 95–108.

[11] P. Djakov, B. Mityagin, Asymptotics of spectral gaps of a Schrödinger operator with a two-term potential, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 351–354.

[12] P. Djakov, B. Mityagin, Asymptotics of the instability zones of Hill operators with two-term potential, OSU Math Research Institute Preprint 04-10, July 2004 (see in: www.math.ohio-state.edu/mri/preprints/).

[13] M.S.P. Eastham, The Spectral Theory of Periodic Differential Operators, Hafner, New York, 1974.

[14] A. González-López, N. Kamran, P. Olver, Quasi-exact solvability, Contemporary Mathematics, vol. 160, American Mathematical Society, 1994, pp. 113–140.

[15] A. Grigis, Estimations asymptotiques des intervalles d'instabilité pour l'équation de Hill, Ann. Sci. École Norm. Sup. 20 (4) (1987) 641–672.

[16] E. Harrell, On the effect of the boundary conditions on the eigenvalues of ordinary differential equations, Amer. J. Math. (suppl) (1981), dedicated to P. Hartman, Baltimore, John Hopkins Press.

[17] E. Hille, On the zeros of Mathieu functions, Proc. London Math. Soc. 23 (1923) 185–237.

[18] H. Hochstadt, Estimates on the stability intervals for the Hill's equation, Proc. Amer. Math. Soc. 14 (1963) 930–932.

[19] C. Hunter, B. Guerrieri, The eigenvalues of Mathieu's equation and their branch points, Stud. Appl. Math. 64 (1981) 113–141.

[20] E.L. Ince, A proof of the impossibility of the coexistence of two Mathieu functions, Proc. Cambridge Philos. Soc. 21 (1922) 117–120.

[21] E.L. Ince, A linear differential equation with periodic coefficients, Proc. Lond. Math. Soc. 23 (1923) 56–74.

[22] E.L. Ince, The real zeros of solutions of a linear differential equation with periodic coefficients, Proc. Lond. Math. Soc. 25 (1926) 53–58.

[23] K. Klotter, G. Kotowski, Uber die Stabilitat der Losungen Hillscher Differentialgleichen mit drei unabhangigan Parametern, Z. Angew. Math. Mech. 23 (1943) 149–155.

[24] B.M. Levitan, Sargsian, Introduction to spectral theory; Selfadjoint ordinary differential operators, Translation of Mathematical Monography, vol. 39, American Mathematical Society, Providence, RI, 1975.

[25] D.M. Levy, J.B. Keller, Instability intervals of Hill's equation, Comm. Pure Appl. Math. 16 (1963) 469–476.

[26] W. Magnus, S. Winkler, Hill's Equation, Interscience Publishers, Wiley, 1969.

[27] V.A. Marchenko, Sturm–Liouville operators and applications, Oper. Theory Adv. Appl., vol. 22, Birkhäuser, Basel, 1986.

[28] Z. Markovic, On the impossibility of simultaneous existence of two Mathieu functions, Proc. Cambridge Philos. Soc. 23 (1926) 203–205.

[29] N.W. McLachlen, Theory and Applications of Mathieu Functions, Oxford University Press, Oxford, 1947.

[30] J. Meixner, F.W. Schäfke, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954.

[31] J. Meixner, F.W. Schäfke, G. Wolf, Mathieu functions and spheroidal functions and their mathematical foundations, Lecture Notes in Mathematics, vol. 837, Springer, Berlin, 1980.

[32] J. Pöschel, E. Trubowitz, Inverse Spectral Theory, Academic Press, New York, 1987.

[33] M. Razavy, An exactly soluble Schrödinger equation with a bistable potential, Amer. J. Phys. 48 (1980) 285 –288.

[34] M.A. Shifman, A. Turbiner, Quantum problems with partial algebraization of spectra, Comm. Math. Phys. 126 (1989) 347–365.

[35] B. Simon, Coupling constant analyticity for the anharmonic oscillator, Ann. Phys. 58 (1970) 76–136.

[36] A. Sommerfeld, Wave Mechanics, New York, 1929.

[37] A. Turbiner, Spectral Riemannian surfaces of the Strurm–Liouville operator and quasi-exactly-solvable problems, Funktz. Analiz i ego Prilozhenia 22 (1988) 92–94 (Russian); Soviet Math. Funct. Anal. Appl. 22 (1988) 163–166 (Engl. transl.).

[38] A. Turbiner, Quantum mechanics: the problems lying between exactly-solvable and non-solvable, Sov. Phys. ZhETF 94 (1988) 33–45 Engl. transl.: JETP 67 (1988) 230–236.
[39] A. Turbiner, Lie Algebras and linear operators with invariant subspace, in: N. Kamran, P. Olver (Eds.), Lie Algebras, Cohomologies and New Findings in Quantum Mechanics, Contemporary Mathematics, vol. 160, American Mathematical Society, 1994, pp. 263–310.
[40] A. Turbiner, Quasi-exactly-solvable differential equations, in: N.H. Ibragimov (Ed.), [CRC Handbook of] Lie Group Analysis of Differential Equations, vol. 3, New Trends in Theoretical Developments and Computational Methods, CRC Press, Boca Raton - New York - London - Tokyo, 1996, pp. 329–364.
[41] K.M. Urwin, F.M. Arscott, Theory of the Whittaker–Hill equation, Proc. Roy. Soc. Edinburgh Sect. A 69 (1970/71) 28–44.
[42] H. Volkmer, Quadratic growth of convergence radii for eigenvalues of two-parameter Sturm–Liouville equations, J. Differential Equations 128 (1996) 327–345.
[43] H. Volkmer, Convergence radii for eigenvalues of two-parameter Sturm–Liouville problems, Analysis (Munich) 20 (3) (2000) 225–236.
[44] H. Volkmer, Coexistence of periodic solutions of Ince's equation, Analysis (Munich) 23 (2003) 97–105.
[45] H. Volkmer, On Riemann surfaces of analytic eigenvalue functions, Complex Var. Theory Appl. 49 (2004) 169–182.
[46] J. Weidmann, Spectral theory of ordinary differential operators, Lecture Notes in Mathematics, vol. 1258, Springer, Berlin, 1987.
[47] E.T. Whittaker, On a class of differential equations whose solutions satisfy integral equations, Proc. Edinb. Math. Soc. 33 (1914) 14–23.
[48] S. Winkler, W. Magnus, The coexistence problem for Hill's equation, Research Report No. BR - 26, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, July 1958, pp. 1–91.